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OPEN MAPS, COLIMITS, AND A CONVENIENT CATEGORY OF FIBRE SPACES

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We show that pulling back along an open map preserves all colimits in the category of weak Hausdorff k-spaces. We also show that the category of open maps over a weak Hausdorff k-space is a convenient category of fibre spaces.

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closed category	k-space
convenient category	open map
enriched category	partial map
exponential law	preservation of colimits
ex-space	pullback
fibre space	sequential space
fibred mapping space	weak Hausdorff

Introduction

Using Booth and Brown's definition of the space of partial maps with closed domain [7], we discuss the preservation of colimits by pullbacks and introduce a convenient category of spaces over a fixed base space B. Our results on the preservation of colimits are a prerequisite for the study of the Thom spectrum associated to a map $f: X \to BF$ [23].

Since Steenrod's paper on compactly generated spaces [29], the utility of convenient categories of topological spaces has been apparent. The usual criteria for such a category are that it contain all the spaces of real interest, that it have all limits and colimits, and that it be cartesian closed (that is, for each space X, there is an exponential functor $?^X$ right adjoint to the product functor $? \times X$). Several such categories of spaces are known—the most familiar being sequential spaces [13, 14, 15, 18, 22] and k-spaces [22, 29, 30, 31]. The situation has not been nearly as satisfactory with respect to categories of spaces over a fixed base space B. The category of all spaces over B has limits and colimits but lacks exponents. On the other hand, exponents may be obtained by restricting to bundles [6] or fibrations [4, 7], but at the cost of losing some limits and colimits. Booth and Brown [3, 7] and Day [10] obtain a convenient category of spaces over a base space B by imposing on B a point-separation condition which the allowed spaces over B can not always satisfy. This unbalanced situation is unacceptable in our applications and we remove the difficulty by showing that the category of open maps from compactly generated spaces into a compactly generated base space is a convenient category. This category is sufficiently large because bundles over any space and (Hurewicz) fibrations over well-behaved spaces are open maps.

In Section 1, we introduce Booth and Brown's space of partial maps and relate it to the existence of exponents and the preservation of colimits. These results are applied in Section 2 to show that the category of open maps into a fixed base space is a convenient category. In Section 3, we derive from this category a convenient category of ex-spaces. The fourth section contains a collection of results on the preservation of open maps by standard constructions in topology. Section 5 is devoted to extensions of our results to other categories of spaces.

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1. Partial maps, exponents and colimits

To develop a convenient category of spaces over a base space, we must first choose a convenient category of spaces. We work with two—the category $\mathcal{K}$ of k-spaces and its full subcategory $\mathcal{U}$ of compactly generated spaces [22]. Recall that a subset V of a space X is compactly open (closed) if its inverse image $g^{-1}(V)$ is open (closed) for any map $g : K \to X$ from a compact (Hausdorff) space K into X. A space X is a k-space if its compactly open sets are open—or equivalently, its compactly closed sets are closed. A space X is weak Hausdorff if the continuous image in X of any compact space is closed [27]. Compactly generated spaces are weak Hausdorff k-spaces.

Ideally, we would like to construct exponents for some category of spaces over an arbitrary k-space B; however, this does not seem to be possible. Thus, we always assume that our base space B is in $\mathcal{U}$. We introduce $\mathcal{K}$ solely because our initial constructions must be carried out there.

Let $\mathcal{K}/B$ and $\mathcal{U}/B$ be the categories of k-spaces and compactly generated spaces over B. The objects of $\mathcal{K}/B$ (or $\mathcal{U}/B$) are maps $p : X \to B$ with X in $\mathcal{K}$ (or $\mathcal{U}$) and the morphisms $\lambda : (p : X \to B) \to (q : Y \to B)$ are maps $\lambda : X \to Y$ such that $p = q\lambda$. Any map $q: Y \to B$ in $\mathcal{U}$ induces pullback functors

$$q^*: \mathcal{K}/B \to \mathcal{K}/Y,$$

$$q^*: \mathcal{U}/B \to \mathcal{U}/Y,$$

which take $p: X \to B$ to the pullback $q^*p: q^*X \to Y$ of p along q. In addition to constructing exponents for a category of spaces over B, we wish to show that the functors q^* preserve colimits by constructing right adjoints for them.

The categories $\mathcal{K}/B$ and $\mathcal{U}/B$ have all colimits and limits. Their colimits and equalizers consist of the corresponding constructions in $\mathcal{K}$ or $\mathcal{U}$ together with the induced map to B. Their products are obtained by taking the subspace of the corresponding product in $\mathcal{K}$ or $\mathcal{U}$ on which all the projections to B agree. Recall that colimits in $\mathcal{K}$ are constructed exactly as in the category of all spaces and that colimits in $\mathcal{U}$ are just the maximal weak Hausdorff quotients of the corresponding colimits in $\mathcal{K}$ [16, 22]. Limits in both $\mathcal{K}$ and $\mathcal{U}$ are produced by applying a functor k to the corresponding limits in the category of all spaces; this functor is the right adjoint to the inclusion of k-spaces into all spaces and of compactly generated spaces into weak Hausdorff spaces [22, 30]. Note that, because it is a limit, the product $X \times Y$ of two spaces in $\mathcal{K}$ or $\mathcal{U}$ need not have the usual cartesian product topology; however, it will have this topology if at least one of the two spaces is compact. The nonstandard topology on products is the reason for our use of weak Hausdorff (rather than Hausdorff) spaces; a k-space X is weak Hausdorff if and only if the image of the diagonal $\Delta: X \to X \times X$ is closed in the k-space product topology.

Topos theorists have taught us that the existence of right adjoints for pullback functors and of exponents for the category of objects over a base object is related to the existence of a classifying object for partial maps [21, p. 18, 28]. Booth and Brown have introduced such a partial map classifier for maps with closed domain [7].

Definition 1.1. Let Y be a k-space. Then the partial map classifier $\tilde{Y}$ is the space whose underlying set is the union of Y and a disjoint point ω and whose closed subjects are $\tilde{Y}$ and the closed subsets of Y regarded as subsets of $\tilde{Y}$. The space $\tilde{Y}$ is a k-space and the assignment of $\tilde{Y}$ to Y is functorial. Unfortunately, $\tilde{Y}$ cannot be weak Hausdorff (unless Y is empty) because the point ω is not closed. For any k-space X, there is a one-to-one correspondence between maps $h: X \to \tilde{Y}$ and maps $h: A \to Y$ where A runs over the closed subsets of X. The map $\tilde{h}$ takes $X - A$ to ω and A to $Y \subset \tilde{Y}$ by h. We refer to such a map $h: A \to Y$ as a partial map with closed domain.

If $q: Y \to B$ is a map from a k-space Y, then

$$\mathrm{Graph}(q) = \{(y, q(y)) \in Y \times B\}$$

is a closed subset of $Y \times B$ because it is the inverse image of the diagonal in $B \times B$. This is precisely what we need to define exponents for $\mathcal{K}/B$ (see [4, 7]) and the adjoint for q^*.

Definition 1.2. (i) Let $p: X \to B$ and $q: Y \to B$ be in $\mathcal{K}/B$. The exponent $p^q: X^q \to B$ is defined by the pullback diagram

$$\begin{tikzcd} X^q \arrow[r] \arrow[d, "p^q"'] & \tilde{X}^Y \arrow[d, "\tilde{p}^Y"] \\ B \arrow[r, "\theta"] & \tilde{B}^Y \end{tikzcd}$$

where $\tilde{X}^Y$ is the function space in $\mathcal{K}$ and θ is the adjoint to the map

$$B \times Y \to \tilde{B}$$

corresponding to the projection

$$\operatorname{Graph}(q) \subset B \times Y \to B.$$

If we write X_b for $p^{-1}(b)$ when $b \in B$, then $(X^q)_b$ is just the space of maps from Y_b to X_b.

By universality of pullbacks, the construction $p^q: X^q \to B$ is functorial in both p and q.

(ii) Let $q: Y \to B$ be in $\mathcal{U}/B$ and $t: W \to Y$ be in $\mathcal{K}/Y$ (We assume Y is in $\mathcal{U}$ to maintain our convention on base spaces). Define

$$\Pi_q t: \Pi_q W \to B$$

by the pullback diagram

$$\begin{tikzcd} \Pi_q W \arrow[r] \arrow[d, "\Pi_q t"'] & \tilde{W}^Y \arrow[d, "\tilde{t}^Y"] \\ B \arrow[r, "\psi"] & \tilde{Y}^Y \end{tikzcd}$$

where ψ is adjoint to the map

$$B \times Y \to \tilde{Y}$$

corresponding to the projection

$$\operatorname{Graph}(q) \subset B \times Y \to Y.$$

For any b in B, $(\Pi_q W)_b$ is the space of sections of $t^{-1} Y_b \xrightarrow{t} Y_b$. This construction is functorial in t.

The essential properties of p^q and $\Pi_q t$ follow immediately from the definition of pullbacks.

Proposition 1.3. (i) *Let $p: X \to B$, $q: Y \to B$, and $r: Z \to B$ be in $\mathcal{K}/B$. Then there is a natural isomorphism*

$$\mathcal{K}/B(r \times_B q, p) \cong \mathcal{K}/B(r, p^q)$$

where $r \times_B q : Z \times_B Y \to B$ *is the fibre product of* r *and* q *(the categorical product in* $\mathcal{K}/B$*). Thus* $?^q$ *is right adjoint to* $? \times_B q$ *and* $\mathcal{K}/B$ *is a cartesian closed category.*

(ii) *Let* $p : X \to B$ *be in* $\mathcal{K}/B$, $q : Y \to B$ *be in* $\mathcal{U}/B$ *and let* $t : W \to Y$ *be in* $\mathcal{K}/Y$. *Then there is a natural isomorphism*

$$\mathcal{K}/Y(q^*p, t) \cong \mathcal{K}/B(p, \Pi_q t)$$

so that Π_q *is right adjoint to* q^*. *Thus,* $q^* : \mathcal{K}/B \to \mathcal{K}/Y$ *preserves colimits.*

From our description of colimits in $\mathcal{U}$, we obtain a weak result on the behavior of $q^* : \mathcal{U}/B \to \mathcal{U}/Y$ with respect to colimits.

Corollary 1.4. *For any* $q : Y \to B$ *in* $\mathcal{U}/B$, *the functor*

$$q^* : \mathcal{U}/B \to \mathcal{U}/Y$$

preserves any colimit in $\mathcal{U}/B$ *whose total space is not a proper quotient of the total space of the corresponding colimit in* $\mathcal{K}/B$.

Proposition 1.3 (i) is due to Booth [3, 7] and Day [10]; Proposition 1.3 (ii) is due to Booth and Brown [7]. These results are unsatisfactory for our purposes because, even though they require the base space to be weak Hausdorff, they can not insure that the total spaces are also weak Hausdorff. To obtain a better result, we must restrict attention to open maps—that is, maps which take open sets to open sets.

Proposition 1.5. *Let* $q : Y \to B$ *be in* $\mathcal{U}/B$. *Then the following are equivalent.*

(i) q *is open.*

(ii) $q^* : \mathcal{U}/B \to \mathcal{U}/Y$ *preserves all colimits.*

(iii) $? \times_B q : \mathcal{U}/B \to \mathcal{U}/B$ *preserves all colimits.*

(iv) *For any* $t : W \to Y$ *in* $\mathcal{U}/Y$, *the total space* $\Pi_q W$ *of* $\Pi_q t$ *is weak Hausdorff so the functor* $\Pi_q : \mathcal{K}/Y \to \mathcal{K}/B$ *restricts to a functor*

$$\Pi_q : \mathcal{U}/Y \to \mathcal{U}/B$$

right adjoint to $q^* : \mathcal{U}/B \to \mathcal{U}/Y$.

(v) *For any* $p : X \to B$ *in* $\mathcal{U}/B$, *the total space* X^q *of* p^q *is weak Hausdorff so that* $?^q : \mathcal{K}/B \to \mathcal{K}/B$ *restricts to a functor*

$$?^q : \mathcal{U}/B \to \mathcal{U}/B$$

right adjoint to $? \times_B q : \mathcal{U}/B \to \mathcal{U}/B$.

To prove this proposition, we need a lemma whose proof would be trivial if we were not using the non-standard topology on products and pullbacks requred in $\mathcal{U}$.

Lemma 1.6. (i) *For any spaces* X *and* Y *in* $\mathcal{U}$, *the projection map* $\pi_1 : Y \times X \to Y$ *is an open map.*

(ii) *If $p: X \to B$ is an open map in $\mathcal{U}$ and $q: Y \to B$ is any map in $\mathcal{U}$, then the pullback map $q^*p: q^*X \to Y$ is an open map.*

Proof. It suffices to show that π_1 and q^*p take open sets to compactly open sets. From this, it follows that we may assume that Y is compact. When Y is compact, $Y \times X$ and $Y \times B$ have the usual cartesian product topologies and (i) follows from the standard result for the topology. For (ii), note that $Y \times X \xrightarrow{1 \times p} Y \times B$ is open when p is open and Y is compact, and that the pullbacks in the two diagrams

$$\begin{tikzcd} & X \arrow[d, "p"] \\ Y \arrow[r, "q"] & B \end{tikzcd} \qquad \begin{tikzcd} & Y \times X \arrow[d, "1 \times p"] \\ Y \arrow[r, "{(1,q)}"] & Y \times B \end{tikzcd}$$

are the same. Using the fact that $(1, q)$ is injective, it is easy to see that $(1, q)^*(1 \times p)$ is open.

Proof of 1.5. That (v) implies (iii) and (iv) implies (ii) are standard (see [24, p. 114]). We show that (ii) or (iii) implies (i) and (i) implies (iv) and (v) to complete the proof.

Any functor which preserves colimits must preserve epi maps because these can be described in terms of colimits [24, p. 72]. Thus, to show that (ii) or (iii) implies (i), it suffices to show that if q is not open, then neither q^* nor $? \times_B q$ preserves epi maps. Assume that q is not open and let U be an open set of Y such that $q(U)$ is not open. Let C be the complement of $q(U)$ in B and let D be the closure of C in B. The inclusion $j: C \to D$ has dense image and is therefore epi in $\mathcal{U}$ [8]. The pullback $q^*j: q^*C \to q^*D$ (which is also $j \times_B 1: C \times_B Y \to D \times_B Y$) is not epi because it does not have dense image.

To see that (i) implies (iv) and (v), note that Lemma 1.6 implies that the maps π_1 and π_2 in the pullback diagrams

$$\begin{tikzcd} q^*((\Pi_q W) \times_B (\Pi_q W)) \arrow[r] \arrow[d, "\pi_1"'] & Y \arrow[d, "q"] \\ (\Pi_q W) \times_B (\Pi_q W) \arrow[r] & B \end{tikzcd}$$

and

$$\begin{tikzcd} X^q \times_B X^q \times_B Y \arrow[r] \arrow[d, "\pi_2"'] & Y \arrow[d, "q"] \\ X^q \times_B X^q \arrow[r] & B \end{tikzcd}$$

are open. As instances of the counits of our adjunctions, we have maps

$$\varepsilon_1 : q^*((\Pi_q W) \times_B (\Pi_q W)) \cong q^*\Pi_q(W \times_Y W) \to W \times_Y W,$$

$$\varepsilon_2 : X^q \times_B X^q \times_B Y \cong (X \times_B X)^q \times_B Y \to X \times_B X,$$

which are, essentially, fibrewise evaluation maps. The homeomorphisms appearing in ε_1 and ε_2 come from the preservation of products by right adjoints. The composites $\pi_1(\varepsilon_1)^{-1}$ and $\pi_2(\varepsilon_2)^{-1}$ transform the complements of the diagonals in $W \times_Y W$ and $X \times_B X$ (which are open) into the complements of the diagonals in $(\Pi_q W) \times_B (\Pi_q W)$ and $X^q \times_B X^q$. Thus, the diagonals in these two spaces are closed, and these spaces being closed in $(\Pi_q W) \times (\Pi_q W)$ and $X^q \times X^q$, we have that $\Pi_q W$ and X^q are weak Hausdorff.

Remark 1.7. Our proof of 1.5 does not exhibit a pair p and q in $\mathcal{U}/B$ for which p^q is not in $\mathcal{U}/B$. This leaves open the possibility that the misbehavior of p^q is an exotic phenomenon. To see that this is not so, take B to be the unit interval I and X and Y to be I^+, the union of I and a disjoint basepoint. Let p and q to be the identity on $I \subset I^+$ and take $+$ to 0. Then the total space $(I^+)^q$ of p^q is not weak Hausdorff. To check this, first note that the spaces from which $(I^+)^q$ is constructed are all first countable. Thus, the functor k is not required in the construction, $(I^+)^q$ is first countable, and it is weak Hausdorff if and only if it is Hausdorff. The topology on $(I^+)^q$ has a subbasis consisting of the inverse images of the open sets of $B = I$ and the open sets of $(\tilde{I}^+)^{I^+}$. Let

$$\lambda, \lambda' : (I^+)_0 \to (I^+)_0 = \{0, +\}$$

be the identity map and the constant map at 0 respectively. One can easily check that λ and λ' do not have disjoint open neighborhoods in $(I^+)^q$.

2. The convenient category of open maps

Proposition 1.5 suggests that if we want a convenient category of spaces over B, then we should consider the full subcategory $\mathcal{O}(B)$ of $\mathcal{U}/B$ consisting of the open maps into B. The category $\mathcal{O}(B)$ has all colimits; the colimit in $\mathcal{U}/B$ of a diagram in $\mathcal{O}(B)$ is an open map and so the colimit in $\mathcal{O}(B)$. Limits and exponents for $\mathcal{O}(B)$ cannot be obtained so directly; they are provided by an appropriate right adjoint.

Proposition 2.1. *For any B in $\mathcal{U}$, the inclusion functor from $\mathcal{O}(B)$ to $\mathcal{U}/B$ has a right adjoint*

$$O : \mathcal{U}/B \to \mathcal{O}(B).$$

Proof. For any $p : X \to B$ in $\mathcal{U}/B$, let $OX \subset X$ be the union of all the subspaces A of X such that $(p|A) : A \to B$ is open. Then $Op = (p|OX) : OX \to B$ is an open map.

Of course, OX may be the empty space, but the inclusion of the empty subspace is an open map. If $\lambda : (q : Y \to B) \to (p : X \to B)$ is a morphism in $\mathcal{U}/B$ from an open map q, then the restriction of p to the image of $\lambda : Y \to X$ is an open map. Thus, $\lambda : Y \to X$ factors uniquely through $OX \subset X$. This provides the required natural isomorphism

$$\mathcal{U}/B(q, p) \cong \mathcal{O}(B)(q, Op).$$

Corollary 2.2. (i) *The category $\mathcal{O}(B)$ has all limits; limits in $\mathcal{O}(B)$ are formed by applying O to the corresponding limits in $\mathcal{U}/B$.*

(ii) *The category $\mathcal{O}(B)$ is cartesian closed. For any $q: Y \to B$ in $\mathcal{O}(B)$, the adjoint to the product functor*

$$? \times_B q : \mathcal{O}(B) \to \mathcal{O}(B)$$

is

$$O(?^q) : \mathcal{O}(B) \to \mathcal{O}(B).$$

(iii) *For any $q: Y \to B$ in $\mathcal{O}(B)$, the pullback functor*

$$q^* : \mathcal{O}(B) \to \mathcal{O}(Y)$$

has right adjoint

$$O\Pi_q : \mathcal{O}(Y) \to \mathcal{O}(B).$$

Remark 2.3. (i) Corollary 2.2 and the remarks preceding it on colimits in $\mathcal{O}(B)$ complete our proof that $\mathcal{O}(B)$ is a convenient category of spaces over B.

(ii) The proof of 2.2 (ii) and the statement of 2.2 (iii) both assume Lemma 1.6 (ii) which asserts that the pullback of an open map is an open map. This insures that q^* restricts to a functor from $\mathcal{O}(B)$ to $\mathcal{O}(Y)$ and that the fibre product $p \times_B q$ is in $\mathcal{O}(B)$ if p and q are in $\mathcal{O}(B)$.

(iii) The functor O is not well understood and is best dealt with by showing that it is unnecessary in the cases of greatest interest. We have already noted that it is not required for finite products. Also, if p and q are bundles, the p^q is open, being also a bundle [6], and Op^q is p^q. Similarly, if p and q are (Hurewicz) fibrations, then so is p^q by [4]. If B is locally equiconnected (that is, the diagonal $\Delta : B \to B \times B$ is a cofibration), then fibrations over B, being submersions [7, Theorem 6.2], are open maps and we do not need O to form exponents for fibrations. Nevertheless, O is sometimes necessary for exponents. Let $p: U \to B$ and $q: V \to B$ be the inclusions of open subsets U and V of B with V not also closed. It is easy to check that p^q is the inclusion of the non-open subspace $B - V$ into B [7, Ex. 4.3]. Of course, Op^q is then the inclusion of the interior of $B - V$ into B.

Remark 2.4. Our ultimate objective is to do homotopy theory in the category $\mathcal{O}(B)$. For this, we need one additional observation best explained by an analogy with homotopy theory in $\mathcal{U}$. A homotopy λ between two maps λ_0, $\lambda_1 : X \to Y$ in $\mathcal{U}$ can be described in any one of three equivalent forms

$$\lambda : I \to Y^X, \qquad \lambda : X \times I \to Y, \qquad \lambda : X \to Y^I.$$

The first description translates easily to $\mathcal{O}(B)$. The set $\mathcal{O}(B)(p, q)$ of morphisms from $p : X \to B$ to $q : Y \to B$ can be topologized as a subspace of Y^X and a homotopy between two morphisms λ_0, $\lambda_1 : p \to q$ in $\mathcal{O}(B)$ is just a continuous map

$$\lambda : I \to \mathcal{O}(B)(p, q).$$

To translate the other two descriptions, we must define $p \times I$ and q^I. For any space Z in $\mathcal{U}$ and any $p : X \to B$ in $\mathcal{O}(B)$, let $p \times Z$ be the composite

$$p \times Z : X \times Z \xrightarrow{\pi_1} X \xrightarrow{p} B,$$

where π_1 is the projection; by Lemma 1.6, $p \times Z$ is in $\mathcal{O}(B)$. Let p^Z be the exponent $O(p^{\pi_1})$ where $\pi_1 : B \times Z \to B$ is the projection. There are natural isomorphisms

$$\mathcal{O}(B)(p \times Z, q) \cong \mathcal{U}(Z, \mathcal{O}(B)(p, q)) \cong \mathcal{O}(B)(p, q^Z).$$

Taking Z to be I, we obtain our three equivalent descriptions of a homotopy.

Category theorists describe the structure just displayed on $\mathcal{O}(B)$ by saying that $\mathcal{O}(B)$ is enriched over $\mathcal{U}$ (via the topology on the sets $\mathcal{O}(B)(p, q)$) and that it has tensors (the objects $p \times Z$) and cotensors (the objects p^Z). See [11, 12, 20] for the definitions and basic results. It is a purely formal, but somewhat arduous, exercise to show that all the basic concepts and tools of homotopy theory—such as cofibrations, fibrations, homotopy limits and colimits, Barratt-Puppe sequences, and Milnor $\lim^1$ sequences—are available in any category enriched over $\mathcal{U}$ which has tensors, cotensors, colimits and limits.

3. A convenient category of ex-spaces

Since a major portion of the machinery of homotopy theory involves based spaces and based maps, an important aspect of setting up this machinery for a category of spaces over B involves the introduction of the analog of based spaces. Following James [17], we call these objects ex-spaces. Here, we introduce a convenient category of open ex-spaces, the "based" category corresponding to $\mathcal{O}(B)$. This category should be useful in the study of ex-spectra, and thus of transfer maps for fibrations [1, 9].

An ex-space $p : X \to B$ consists of a total space X and a base space B (both in $\mathcal{U}$) together with a projection $p : X \to B$ and a section $s : B \to X$ such that $ps = 1$. We think of s as just providing basepoints to the fibres X_b of p (for $b \in B$); hence, we denote sections generically by s and suppress any mention of them from our notation. An open ex-space is one for which the projection is an open map. A map of ex-spaces

$\lambda:(p:X\to B)\to(q:Y\to B)$ is a map $\lambda:X\to Y$ such that $p=q\lambda$ and $s=\lambda s$. We denote the category of ex-spaces over B by $\mathrm{Ex}(B)$ and the subcategory of open ex-spaces by $\mathcal{O}_*(B)$. Note that the functor $O:\mathcal{U}/B\to\mathcal{O}(B)$ of Proposition 2.1 provides a right adjoint $O:\mathrm{Ex}(B)\to\mathcal{O}_*(B)$ to the inclusion functor.

The categories $\mathrm{Ex}(B)$ and $\mathcal{O}_*(B)$ have all limits and colimits. Limits in $\mathrm{Ex}(B)$ are formed by taking the corresponding limit in $\mathcal{U}/B$ together with the obvious section; limits in $\mathcal{O}_*(B)$ are obtained from those in $\mathrm{Ex}(B)$ via the functor O. Colimits in both $\mathrm{Ex}(B)$ and $\mathcal{O}_*(B)$ are obtained from the corresponding colimits in $\mathcal{U}/B$ by identifying the images of all the sections. Thus, the coproduct of a pair of (open) ex-spaces $p:X\to B$ and $q:Y\to B$ is the fibrewise wedge product $p\vee q:X\vee Y\to B$; the fibre $(p\vee q)_b$ is just the ordinary wedge product $X_b\vee Y_b$.

Just as the closed structure on the category $\mathcal{T}$ of based compactly generated spaces comes from the smash product, so the closed structure on $\mathcal{O}_*(B)$ comes from the fibrewise smash product. For ex-spaces $p:X\to B$ and $q:Y\to B$, the ex-space $p\wedge q:X\wedge Y\to B$ is obtained by fibrewise collapsing the image of $p\vee q$ in $p\times_B q$. The fibre $(X\wedge Y)_b$ is just the ordinary smash product $X_b\wedge Y_b$. Because $p\wedge q$ is formed as a colimit, $p\wedge q$ is open if p and q are. Since the points of the total space Z^q of the exponent p^q of section one are maps $Y_b\to X_b$, we can define an ex-space

$$F(q,p):F(q,X)\to B$$

for open ex-spaces p and q by letting $F(q,X)$ be the subspace of X^q consisting of the basepoint preserving maps. The section $B\to F(q,X)$ takes $b\in B$ to the trivial map $Y_b\to X_b$. The adjunction between $?\wedge q$ and $F(q,?)$ follows easily from the analogous one between $?\times_B q$ and $?^q$ (see Proposition 1.3 (i) and Corollary 2.2).

Proposition 3.1. *Let $p:X\to B$, $q:Y\to B$ and $r:Z\to B$ be in $\mathcal{O}_*(B)$. Then there is a natural isomorphism*

$$\mathcal{O}_*(B)(r\wedge q,p)\cong\mathcal{O}_*(B)(r,OF(q,p)).$$

For any $q:Y\to B$ in $\mathcal{U}/B$, we obtain a pullback functor $q^*:\mathrm{Ex}(B)\to\mathrm{Ex}(Y)$, and this restricts to a functor

$$q^*:\mathcal{O}_*(B)\to\mathcal{O}_*(Y).$$

Moreover, when $t:W\to Y$ is an ex-space, the map $\Pi_q t:\Pi_q W\to B$ of Definition 1.2 (ii) is an ex-space; the fibre $(\Pi_q W)_b$ is the space of sections of $t^{-1}Y_b\to^t Y_b$ with basepoint the restriction to Y_b of the canonical section $s:Y\to W$ of t. Again, the adjunctions of sections one and two convert into an adjunction for open ex-spaces.

Proposition 3.2. *Let $q:Y\to B$ be an open map. Then the pullback functor*

$$q^*:\mathcal{O}_*(B)\to\mathcal{O}_*(Y)$$

has a right adjoint

$$O\Pi_q:\mathcal{O}_*(Y)\to\mathcal{O}_*(B).$$

Remark 3.3. As in Remark 2.4, we can topologize the set $\mathcal{O}_*(B)(q,p)$ of maps between open ex-spaces $p: X \to B$ and $q: Y \to B$ as a subspace of X^Y. The space $\mathcal{O}_*(B)(q,p)$ has the trivial map from q to p as a basepoint. Moreover, for any based space Z, we can define open ex-spaces

$$p \wedge Z : X \wedge Z \to B, \qquad F(Z,p) : F_B(Z,X) \to B$$

such that for any b in B, $(X \wedge Z)_b$ is the ordinary smash product $X_b \wedge Z$ and $F_B(Z,X)_b$ is, roughly speaking, the space $F(Z, X_b)$ of based maps from Z to X_b (roughly because we must apply O). The easiest way to define these is to note that the projection $\pi_1 : B \times Z \to B$ is an open ex-space whose section takes b in B to $(b, *)$, where $*$ is the basepoint of Z. Then $p \wedge Z$ is just $p \wedge \pi_1$ and $F(Z,p)$ is $OF(\pi_1, p)$. As in Remark 2.4, we have natural isomorphisms

$$\mathcal{O}_*(B)(p \wedge Z, q) \cong \mathcal{T}(Z, \mathcal{O}_*(B)(p,q)) \cong \mathcal{O}_*(B)(p, F(Z,q))$$

for p and q in $\mathcal{O}_*(B)$ and Z in $\mathcal{T}$. Thus, $\mathcal{O}_*(B)$ is enriched over the category $\mathcal{T}$ of based spaces and has tensors and cotensors.

Specializing Z to S^1, I or I^+, we obtain the fibrewise suspension $\Sigma p = p \wedge S^1$ and loop space $\Omega p = F(S^1, p)$, and cone $Cp = p \wedge I$ and path space $Pp = F(I,p)$, and the cylinder $p \wedge I^+$ and free path space $F(I^+, p)$. Of course, $p \wedge I^+$ and $F(I^+, p)$ are used to define homotopies in $\mathcal{O}_*(B)$.

4. The preservation of open maps

Having shown that open maps deserve special attention, we now discuss the preservation of them under such topological constructions as pushouts, colimits of directed systems and the geometric realization of simplicial spaces.

In the study of the relationship between open maps and colimits, we frequently encounter commuting squares

$$\begin{array}{ccc} W & \xrightarrow{F} & Z \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

and subsets U of Z for which we must know that $f^{-1}gU = GF^{-1}U$. A sufficient condition for this is that the natural map $W \to X \times_Y Z$ is surjective; we describe this condition by saying that W maps onto the pullback in the square.

Lemma 4.1. *Let $\{X_\alpha; \lambda^\alpha_\beta : X_\alpha \to X_\beta\}$ and $\{Y_\alpha; \gamma^\alpha_\beta : Y_\alpha \to Y_\beta\}$ be directed systems of injective maps in $\mathcal{U}$ indexed on the same directed set. For each α, let $f_\alpha : Y_\alpha \to X_\alpha$ be* *an open map such that for each $\beta \geq \alpha$, the diagram*

$$\begin{tikzcd}
Y_\alpha \arrow[r, "\gamma^\alpha_\beta"] \arrow[d, "f_\alpha"'] & Y_\beta \arrow[d, "f_\beta"] \\
X_\alpha \arrow[r, "\lambda^\alpha_\beta"] & X_\beta
\end{tikzcd}$$

commutes and Y_α maps onto the pullback. If X and Y are the colimits of the systems and $f: Y \to X$ is the map induced by the f_α, then f is an open map.

Proof. Let $\Lambda_\alpha : X_\alpha \to X$ and $\Gamma_\alpha : Y_\alpha \to Y$ be the natural maps into the colimits. The injectivity of the λ^α_β and γ^α_β implies that the maps Λ_α and Γ_α are injective and that the colimits X and Y in $\mathcal{U}$ are just the colimits in $\mathcal{K}$ (rather than proper quotients of the $\mathcal{K}$-colimits). To show that f is open, it suffices to show that $\Lambda_\alpha^{-1} f(U)$ is open for every α and every open set U of Y. This follows immediately from the easy observation that Y_α maps onto the pullback in the square

$$\begin{tikzcd}
Y_\alpha \arrow[r, "\Gamma_\alpha"] \arrow[d, "f_\alpha"'] & Y \arrow[d, "f"] \\
X_\alpha \arrow[r, "\Lambda_\alpha"] & X
\end{tikzcd}$$

Lemma 4.2. *In the commuting diagram in $\mathcal{U}$ below, assume that j_2 is injective, $(F_1, J_1): X_1 \amalg Y_1 \to P_1$ is surjective, the $A_2X_2Y_2P_2$ face is a pushout and A_1 maps onto the pullbacks in the $A_1A_2X_1X_2$ and $A_1A_2Y_1Y_2$ faces. Then Y_1 and X_1 map onto the pullbacks in the faces $Y_1P_1Y_2P_2$ and $X_1P_1X_2P_2$ respectively. Moreover, if q and r are open, then so is s.*

$$\begin{tikzcd}
 & A_1 \arrow[rr, "j_1"] \arrow[dl, "f_1"'] \arrow[dd] & & X_1 \arrow[dl, "F_1"'] \arrow[dd, "r"] \\
Y_1 \arrow[rr, "J_1"] \arrow[dd, "q"'] & & P_1 \arrow[dd, "s"] & \\
 & A_2 \arrow[rr, "j_2"] \arrow[dl, "f_2"] & & X_2 \arrow[dl] \\
Y_2 \arrow[rr] & & P_2 &
\end{tikzcd}$$

Remark 4.3. In the natural applications of Lemma 4.2, the maps j_1 and j_2 will be closed inclusions and the $A_1X_1Y_1P_1$ face will be a pushout. Under these conditions, it suffices that A_1 maps onto the appropriate pullbacks.

Together, Lemma 4.1 and 4.2 provide a sufficient condition for the preservation of open maps by geometric realization. We use the notation of [26] for simplicial spaces.

Proposition 4.4. *Let $\underset{\sim}{f}: \underset{\sim}{Y} \to \underset{\sim}{X}$ be a map of simplicial spaces such that the maps*

$$f_n : Y_n \to X_n$$

are open (for $n \geq 0$) and such that Y_n maps onto the pullbacks in the squares

$$\begin{tikzcd} Y_n \arrow[r, "s_i"] \arrow[d, "f_n"'] & Y_{n+1} \arrow[d, "f_{n+1}"] \\ X_n \arrow[r, "s_i"] & X_{n+1} \end{tikzcd} \quad (\text{for } n \geq 0, 0 \leq i \leq n)$$

$$\begin{tikzcd} Y_n \arrow[r, "d_i"] \arrow[d, "f_n"'] & Y_{n-1} \arrow[d, "f_{n-1}"] \\ X_n \arrow[r, "d_i"] & X_{n-1} \end{tikzcd} \quad (\text{for } n \geq 1, 0 \leq i \leq n).$$

Then the geometric realization, $|\underset{\sim}{f}| : |\underset{\sim}{Y}| \to |\underset{\sim}{X}|$, of f is an open map.

The conditions of Proposition 4.4 have an easy interpretation in the two-sided geometric bar construction of [25].

Corollary 4.5. *Let G be a topological monoid and let X and Y be left and right G-spaces. If the action map $g: X \to X$ is surjective for each $g \in G$, then the natural map*

$$B(Y, G, X) \to B(Y, G, *)$$

is an open map.

This corollary is the key to applying the results of Section 1 to the study of Thom spectra [23].

5. Applications to other categories

The results of the previous sections extend easily to two categories of spaces other than $\mathscr{U}$. In all of these settings, there is an alternative to $\mathcal{O}(B)$ as a convenient category of spaces over a base space B.

First, if G is a topological group, then we can replace $\mathscr{U}$ and $\mathscr{T}$ everywhere by the categories $G\mathscr{U}$ and $G\mathscr{T}$ of unbased and based left G-spaces (basepoints are assumed to have trivial G-action). The only observations required are that, for G-spaces X and Y, the function space X^Y must be given the conjugation G-action and that the point ω of the space $\tilde{Y}$ of Definition 1.1 must be given trivial G-action.

Second, the categories $\mathscr{K}$ and $\mathscr{U}$ can be replaced by the categories of sequential spaces and sequential spaces with unique sequential limits [13, 14, 15, 18, 22]. For the alteration, we need only replace the arbitrary compact (Hausdorff) spaces in our proofs by the single space $\{0\} \cup \{1/n \mid n \geq 1\}$ topologized as a subset of the unit interval. One advantage of sequential spaces is that the appropriate topology for X^Y has a simple description. A sequence $\{f_n \mid n \geq 0\}$ converges to f in X^Y if and only if for every convergent sequence $\{y_m \mid m \geq 0\}$ with limit $y \in Y$, the net $\{f_n(y_m) \mid m, n \geq 0\}$ converges to $f(y)$ in X. There does not seem to be an analogous characterization of the function space topologies for $\mathscr{K}$ and $\mathscr{U}$.

For each of these categories of spaces, $\mathcal{O}(B)$ has a full subcategory which is also convenient. As in [7], we say that a map $p: X \to B$ is a submersion if for each x in X, there is an open neighborhood V of $p(x)$ and a map $\lambda: V \to X$ with $\lambda p(x) = x$ and $p\lambda = 1$. Roughly speaking, submersions are maps with enough local sections. Let $\mathrm{Sub}(B)$ be the full subcategory of $\mathscr{U}/B$ consisting of submersions. A submersion is clearly an open map so $\mathrm{Sub}(B)$ is a subcategory of $\mathcal{O}(B)$. Moreover, the proof of Proposition 2.1 extends in an obvious way to provide a right adjoint to the inclusion of $\mathrm{Sub}(B)$ into $\mathscr{U}/B$. Thus, $\mathrm{Sub}(B)$ is a cartesian closed category with all limits and colimits.

Remark 5.1. When working with G-spaces, it may be appropriate to alter the definition of a submersion $p: X \to B$ to say that for each $x \in X$, there is a neighborhood V of the orbit $Gp(x)$ of $p(x)$ and a map $\lambda: V \to X$ such that $\lambda p(x) = x$ and $p\lambda = 1$. It is not appropriate to insist that these local sections be equivariant because, if properly defined, even trivial G-bundles need not have equivariant local sections.

In [19], Johnstone describes a connection between openness and exponentation for locales in the topos of sheaves on a space B which is analogous to our results here.

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